DUALITY SYMMETRY IN THE SCHWARZ-SEN MODEL*

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The duality symmetric but non manifestly covariant action proposed by Schwarz and Sen is canonically quantized in the Coulomb gauge. The resulting theory turns out to be, nevertheless, relativistically invariant. It is shown, afterwards, that the Schwarz-Sen model naturally emerges when duality is implemented as a local symmetry of sourceless electrodynamics. This implies in the equivalence of these theories at the quantum level.

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I. INTRODUCTION

The equations of motion of the four-dimensional low energy effective field theory for the bosonic sector of the heterotic string, are invariant under SL(2,R) duality transformations of the massless fields involved. This, so called S-duality symmetry, is a symmetry of the equations of motion but not of the corresponding action. On the other hand, the low energy effective field theory action retains the target space duality symmetry (T-duality) of string theory. It would be desirable to achieve S-duality at the level of the action in the hope of attaining results similar to those given by the T-duality symmetry. However, it is difficult to conciliate duality symmetry and manifest Lorentz covariance.

The difficulty of writing a non trivial action involving only a finite-component self-dual covariant form is well known. In fact, to construct a duality symmetric action being manifestly covariant one must introduce auxiliary fields. The first attempts in this direction involved an infinite number of auxiliary fields [1].

Duality symmetric actions being manifestly covariant and containing a finite number of auxiliary fields have also been constructed but they are not polynomials. These actions were found by Pasti, Sorokin and Tonin (PST) [2,3]. For the case of electrodynamics, the PST action covariantizes the action found earlier by Deser and Teitelboim [4] and rediscovered by Schwarz and Sen [5], which is duality symmetric but not manifestly covariant. The introduction of sources into the manifestly covariant and non-manifestly covariant versions of these duality symmetric actions has also been achieved [6].

The present work is essentially dedicated to establish the quantum mechanical equivalence of the Schwarz-Sen and Maxwell theories in the case free of sources. We start by quantizing the Schwarz-Sen model in the Coulomb gauge. The resulting quantum theory will be shown to be, nevertheless, relativistically invariant. This is our Section II. In Section III we begin by recalling that the Maxwell action, when formulated in the Coulomb gauge, remains invariant under a set of non-local duality transformations derived by Deser and Teitelboim [4]. Additional fields are, afterwards, brought into the theory in order to make these transformations local. Correspondingly, a new expression for the generating functional of Green functions is derived. In Section IV we demonstrate that the Green functions generating functionals for the Maxwell and Schwarz-Sen theories are rigorously identical. Section V contains the conclusions.

II. QUANTIZATION OF THE SCHWARZ-SEN MODEL

The duality symmetric action proposed by Schwarz-Sen [5] involves two gauge potentials $A^{\mu,a}(0 \leq \mu \leq 3,\ 1 \leq a \leq 2)$ and reads[1]

$$S = -\frac{1}{2}\int d^4x \left(B^{a,i}\epsilon_{ab}E^{b,i} \,+\, B^{a,i}B^{a,i}\right) \quad , \tag{1}$$

where

$$E^{a,i} = -F^{a,0i} = -\left(\partial^0 A^{a,i} - \partial^i A^{a,0}\right) \quad , \tag{2a}$$

$$B^{a,i} = -\frac{1}{2}\epsilon^{ijk}F^a_{jk} = -\epsilon^{ijk}\partial_j A^a_k \quad , \tag{2b}$$

and $1 \leq i,j,k \leq 3$. S is separately invariant under the local gauge transformations

$$A^{a,0} \to A^{a,0} + \Psi^a \quad , \tag{3a}$$

$$A^{a,i} \to A^{a,i} - \partial^i \Lambda^a \quad , \tag{3b}$$

and under the global $SO(2)$ rotations

$$A^{\mu a} \to A'^{\mu a} = A^{\mu a}\cos\theta + \epsilon^{ab}A^{\mu b}\sin\theta \quad . \tag{4}$$

Of course, (4) reduces to the usual discrete duality transformation for $\theta = \pi/2$. However, the Lagrangian density in (1),

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[1]This section is mainly based on Refs. [7,8]. Our space-time metric is $g_{00} = -g_{11} = -g_{22} = 1$ while ϵ_{ab} designates a generic element of the two-dimensional antisymmetric unit matrix ($\epsilon_{12} = +1$).

$$\mathcal{L} = \frac{1}{2}\epsilon^{jki}(\partial_j A^a_k)\epsilon_{ab}(\partial_0 A^b_i) - \frac{1}{2}\epsilon^{jki}(\partial_j A^a_k)\epsilon_{ab}(\partial_i A^b_0) - \frac{1}{4}F^{a,jk}F^a_{jk} \ , \tag{5}$$

is not a Lorentz scalar. The use of the equations of motion deriving from (5),

$$\epsilon^{ijk}\epsilon_{ab}\partial_0\partial_j A^b_k + \partial_j\left(\partial^j A^{a,i} - \partial^i A^{a,j}\right) = 0 \ , \tag{6}$$

allows for the elimination from S of one of the gauge fields, the action for the remaining one being the conventional Maxwell action.

Within the Hamiltonian framework, the Schwarz-Sen model is characterized by the canonical Hamiltonian (H_c)

$$H_c = \int d^3x \left[\frac{1}{2}\epsilon^{jki}(\partial_j A^a_k)\epsilon_{ab}(\partial_i A^b_0) + \frac{1}{4}F^{a,jk}F^a_{jk}\right] \ . \tag{7}$$

Furthermore, the system possesses the primary constraints

$$\Omega^a_0 \equiv \pi^a_0 \approx 0 \ , \tag{8a}$$

$$\Omega^a_i \equiv \pi^a_i + \frac{1}{2}\epsilon_{ab}\,\epsilon_{ijk}\,\partial^j A^{b,k} \approx 0 \ , \tag{8b}$$

where we have designated by π^a_μ the momentum canonically conjugate to $A^{a,\mu}$. Then, the total Hamiltonian (H') is given by $H' = H_c + \int d^3x\left(u^{a,0}\Omega^a_0 + u^{a,i}\Omega^a_i\right)$, where the u's are Lagrange multipliers. Persistence in time of Ω^a_0 produces neither secondary constraints nor determines the Lagrange multipliers. On the other hand, persistence in time of the primary constraints $\{\Omega^a_i\}$ does not lead to the existence of secondary constraints but determines partially the Lagrange multipliers $\{u^a_i\}$. Indeed, since the Poisson bracket

$$[\Omega^a_i(\vec{x})\,,\,\Omega^b_j(\vec{y})]_P = -\epsilon_{ab}\,\epsilon_{ijk}\,\partial^j_x\delta(\vec{x}-\vec{y}) \tag{9}$$

does not vanish, $\dot{\Omega}^a_i = [\Omega^a_i, H']_P \approx 0$ yields $u^{a,i} = \epsilon_{ab}(B^{b,i} - \partial^i\phi^b)$, where ϕ^a is an arbitrary scalar. Thus,

$$\Omega^a(\vec{x}) = \partial^i\Omega^a_i(\vec{x}) \approx 0 \tag{10}$$

and $\Omega^a_0 \approx 0$ are the first-class constraints in the theory.

To isolate the second-class constraints from (8b), we split Ω^a_i into longitudinal (L) and transversal (T) components, namely, $\Omega^a_i = \Omega^a_{Li} + \Omega^a_{Ti}$, where $\Omega^a_{Li} = -\frac{\partial_i\partial^j}{\nabla^2}\Omega^a_j$, $\Omega^a_{Ti} = \left(g^j_i + \frac{\partial_i\partial^j}{\nabla^2}\right)\Omega^a_j$ and $\nabla^2 \equiv -\partial_j\partial^j$. The first-class constraint (10) only involves the longitudinal components Ω^a_{Li} and states that these components vanish individually. Then, the second-class constraints are

$$\Omega^a_{Ti} = \pi^a_{Ti} + \frac{1}{2}\epsilon_{ab}\,\epsilon_{ijk}\,\partial^j A^{b,k}_T \approx 0 \ . \tag{11}$$

The determination of the constraint structure is over. It only remains to be mentioned that the gauge potential $A^{a,\mu}$, when acted upon by the generator of infinitesimal gauge transformations, $G = \int d^3x\,(\Psi^a\Omega^a_0 + \Lambda^a\Omega^a)$, undergoes the change $A^{a,\mu} \to A^{a,\mu} + \delta A^{a,\mu}$ with $\delta A^{a,0} = [A^{a,0}, G]_P = \Psi^a$ and $\delta A^{a,i} = [A^{a,i}, G]_P = -\partial^i\Lambda^a$, in agreement with Eqs.(3).

We shall next quantize the theory by means of the Dirac bracket quantization procedure [9–12]. To this end, we start by fixing the gauge through the subsidiary conditions

$$\chi^{a,0} \equiv A^{a,0} \approx 0 \ , \tag{12a}$$

$$\chi^a \equiv \partial_i A^{a,i} \approx 0 \ . \tag{12b}$$

The fact that the Coulomb condition and $A^{a,0} \approx 0$ are, when acting together, accessible gauge conditions is a peculiarity of the model under analysis. We now recall that, according to the quantization procedure being used, the equal-time commutation algebra is to be abstracted from the corresponding Dirac bracket algebra, the constraint and gauge conditions thereby translating into strong operator relations. After some calculations one finds that

$$\left[A^{a,i}_T(\vec{x})\,,\,A^{b,j}_T(\vec{y})\right] = -i\epsilon_{ab}\,\epsilon^{ijk}\frac{\partial^x_k}{\nabla^2}\delta(\vec{x}-\vec{y}) \ , \tag{13a}$$

$$\left[A^{a,i}_T(\vec{x})\,,\,\pi^b_{Tj}(\vec{y})\right] = \frac{i}{2}\delta_{ab}\left(g^i_j + \frac{\partial^i_x\partial^x_j}{\nabla^2}\right)\delta(\vec{x}-\vec{y}) \ , \tag{13b}$$

$$\left[\pi^a_{Ti}(\vec{x})\,,\,\pi^b_{Tj}(\vec{y})\right] = \frac{i}{4}\epsilon_{ab}\,\epsilon_{ijk}\partial^k_x\delta(\vec{x}-\vec{y}) \ , \tag{13c}$$

while the Hamiltonian operator reads

$$H = \frac{1}{4}\int d^3x F^{a,jk} F^a_{jk} \; = \; -\frac{1}{2}\int d^3x B^{a,j} B^a_j \;\; . \tag{14}$$

One may wonder on whether the right hand side of (14) is afflicted by ordering ambiguities. This not so, since

$$\left[B^{a,i}(\vec{x})\, , \, B^{b,j}(\vec{y})\right] \; = \; i\, \epsilon_{ab}\, \epsilon^{ijk}\, \partial^x_k \delta(\vec{x}-\vec{y}) \;\; . \tag{15}$$

The main object of interest is the field commutator at different space-time points. To find it, we must first solve the Heisenberg equations of motion deriving from (13) and (14), namely,

$$\mathcal{D}^{(-)ab}_{ik} A^{b,k}_T \; = \; 0 \;\; , \tag{16a}$$

$$\partial_0 \pi^a_{Ti} \; = \; \frac{1}{2}\partial^j F^a_{ji} \;\; , \tag{16b}$$

where

$$\mathcal{D}^{(\pm)ab}_{ik} \equiv g_{ik}\delta_{ab}\partial_0 \, \pm \, \epsilon_{ab}\epsilon_{ijk}\partial^j \;\; . \tag{17}$$

Notice that, in the Coulomb gauge, the Lagrange equation of motion (6) can be casted as

$$\epsilon^{jli}\partial_l \, \mathcal{D}^{(-)ab}_{ik} A^{b,k}_T \; = \; 0 \Longrightarrow \mathcal{D}^{(-)ab}_{ik} A^{b,k}_T \; = \; \partial_i \xi^a \;\; . \tag{18}$$

Since $\partial^i \mathcal{D}^{(-)ab}_{ik} A^{b,k}_T \; = \; 0$, the function ξ^a must verify $\nabla^2 \xi^a \; = 0$ but is otherwise arbitrary. Thus, the Lagrangian and the Hamiltonian formulations lead to equivalent equations of motions only after the introduction of a regularity requirement at spatial infinity. This situation resembles that encountered in connection with the theory of the two-dimensional $(x^0, x^1, x^{\pm} \; = \; 1/\sqrt{2}(x^0 \pm x^1))$ self-dual field (Φ) proposed by Floreanini and Jackiw [13,14], where the equations of motion in the Lagrangian and Hamiltonian formulations turn out to be, respectively, $\partial_1\partial_-\Phi = 0$ and $\partial_-\Phi = 0$. We also recall that in order to solve $\partial_-\Phi = 0$ one starts by realizing that $\partial_-\Phi = 0 \Longrightarrow \partial_+\partial_-\Phi = 0 \Longrightarrow \Box\Phi = 0$. The solutions of $\partial_-\Phi = 0$ are then contained in the field of solutions of $\Box\Phi = 0$. We shall follow here a similar approach, since

$$\mathcal{D}^{(-)ab}_{ik} A^{b,k}_T \; = 0 \Longrightarrow \mathcal{D}^{(+)ca,li}\mathcal{D}^{(-)ab}_{ik} A^{b,k}_T \; = \; 0 \Longrightarrow \Box A^{c,l}_T = 0 \;\; . \tag{19}$$

The solving of $\Box A^{a,i}_T = 0$ leads to

$$A^{a,i}_T(x) \; = \; \int d^3y D(x-y) \overleftrightarrow{\partial}^0_y A^{a,i}_T(y) \;\; , \tag{20}$$

where $D(x-y)$ is the zero-mass Pauli-Jordan delta function and $(A \overleftrightarrow{\partial}^k B) \equiv A\partial^k B - B\partial^k A$. The combined use of this last equation and (13) allowed us to find the following explicit form for the field commutator at different space-time points

$$\begin{aligned} &\left[A^{a,i}_T(x)\, , \, A^{b,j}_T(y)\right] \\ &= \; i\left[\delta_{ab}\left(g^{ij} + \frac{\partial^i_x\partial^j_x}{\nabla^2_x}\right) \; - \; \epsilon_{ab}\epsilon^{ijk}\frac{\partial^x_k\partial^x_0}{\nabla^2_x}\right] \; D(x-y) \;\; . \end{aligned} \tag{21}$$

One can verify, by applying $\mathcal{D}^{(-)ca}_{ki}(x)$ to both sides of (21), that the field configurations entering the just mentioned commutator are in fact solutions of (16a).

Now, the function $D(x-y)$ can be given as the sum of a positive plus a negative frequency part and we, therefore, can write

$$A^{a,i}_T(x) \; = \; A^{a,i(+)}_T(x) \, + \, A^{a,i(-)}_T(x) \;\; , \tag{22}$$

where

$$
\begin{aligned}
&A_T^{a,i(\pm)}(x) \\
&= \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2|\vec{k}|}} \exp[\pm i(|\vec{k}|x^0 - \vec{k}\cdot\vec{x})] \sum_{\lambda=1}^{2} \varepsilon_\lambda^{a,i}(\vec{k}) a_\lambda^{(\pm)}(\vec{k})
\end{aligned}
\tag{23}
$$

and $\varepsilon_\lambda^{a,i}(\vec{k}), \lambda = 1, 2$, are unit norm polarization vectors. By going back with (23) into (21) one obtains

$$
\begin{aligned}
&\sum_{\lambda,\lambda'=1}^{2} \varepsilon_\lambda^{a,i}(\vec{k}) \varepsilon_{\lambda'}^{b,j}(\vec{k'}) \left[a_\lambda^{(-)}(\vec{k}) \, , \, a_{\lambda'}^{(+)}(\vec{k'}) \right] \\
&= \left[-\delta_{ab} \left(g^{ij} + \frac{k^i k^j}{|\vec{k}|} \right) + \epsilon_{ab} \epsilon^{ijl} \frac{k_l}{|\vec{k}|} \right] \delta(\vec{k} - \vec{k'}) \ ,
\end{aligned}
\tag{24}
$$

while all others commutators vanish. The polarization vectors are to be found by replacing (23) into the gauge condition (12b) and the equation of motion (16b). One arrives to

$$
\sum_{\lambda=1}^{2} \vec{\varepsilon}_\lambda^{\,a}(\vec{k}) \times \vec{\varepsilon}_\lambda^{\,b}(\vec{k}) = -2 \, \epsilon_{ab} \frac{\vec{k}}{|\vec{k}|} \ . \tag{25}
$$

On the other hand, the Coulomb gauge polarization vectors span, by construction, the space orthogonal to $\vec{k}$, i.e.,

$$
\sum_{\lambda=1}^{2} \varepsilon_\lambda^{a,i}(\vec{k}) \, \varepsilon_\lambda^{a,j}(\vec{k}) = - \left(g^{ij} + \frac{k^i k^j}{|\vec{k}|^2} \right) \ . \tag{26}
$$

By using (25) and (26) we solve at once for the commutator in (24),

$$
\left[a_\lambda^{(-)}(\vec{k}) \, , \, a_{\lambda'}^{(+)}(\vec{k'}) \right] = \delta_{\lambda\lambda'} \, \delta(\vec{k} - \vec{k'}) \ . \tag{27}
$$

Thus the space of states is, as expected, a Fock space with positive definite metric.

Hence, the quantized Schwarz-Sen model is a physically sensible quantum field theory. Our next task is to demonstrate that this theory is also relativistically invariant. We are therefore looking for a set of composite operators $\{\Theta_{\mu\nu}\}$ which may serve as Poincaré densities. We shall build them by following the rules that are used to construct the symmetric (Belinfante) energy-momentum tensor in a manifestly Lorentz invariant theory. In this way we find

$$
\Theta_{00} = -\frac{1}{2} B^{a,i} B_i^a \ , \tag{28a}
$$

$$
\Theta_{0i} = \Theta_{i0} = -\frac{1}{2} \epsilon_{ijk} \, \epsilon_{ab} B^{a,j} \, B^{b,k} \ , \tag{28b}
$$

$$
\Theta_{ij} = \Theta_{ji} = -B_i^a \, B_j^a + g_{ij} \, B^{a,l} \, B_l^a \ . \tag{28c}
$$

Thus, Θ is symmetric and free of ordering ambiguities but we can not yet decide on whether or not it is a tensor. As for the equal-time commutator algebra obeyed by the components of Θ, it is fully determined by the commutator (15). In particular, one can corroborate that

$$
\begin{aligned}
&\left[\Theta^{00}(x^0, \vec{x}) \, , \, \Theta^{00}(x^0, \vec{y}) \right] \\
&= -i \left\{ \Theta^{0k}(x^0, \vec{x}) + \Theta^{0k}(x^0, \vec{y}) \right\} \partial_k^x \delta(\vec{x} - \vec{y}) \ ,
\end{aligned}
\tag{29a}
$$

$$
\begin{aligned}
&\left[\Theta^{00}(x^0, \vec{x}) \, , \, \Theta^{0k}(x^0, \vec{y}) \right] \\
&= -i \left\{ \Theta^{kj}(x^0, \vec{x}) - g^{kj} \, \Theta^{00}(x^0, \vec{y}) \right\} \partial_j^x \, \delta(\vec{x} - \vec{y}) \ ,
\end{aligned}
\tag{29b}
$$

$$
\begin{aligned}
&\left[\Theta^{0k}(x^0, \vec{x}) \, , \, \Theta^{0j}(x^0, \vec{y}) \right] \\
&= i \left\{ \Theta^{0k}(x^0, \vec{y}) \, \partial_x^j + \Theta^{0j}(x^0, \vec{x}) \, \partial_x^k \right\} \delta(\vec{x} - \vec{y}) \ .
\end{aligned}
\tag{29c}
$$

As known [15], positivity requires that a singular Schwinger term, proportional to $(\partial^3)\delta(\vec{x} - \vec{y})$, must also be present in the right hand side of Eq.(29b). The definition of $\Theta^{\mu\nu}$ can, in fact, be altered as to yield such term. But, since

Schwinger terms do not contribute to the algebra of integrated charges, we have omitted them in Eqs.(29). From Eqs.(29) then follows that the charges

$$P^{\mu} \equiv \int d^3x\, \Theta^{0\mu} \ , \tag{30a}$$

$$J^{\mu\nu} \equiv \int d^3x \left(\Theta^{0\mu}x^{\nu} - \Theta^{0\nu}x^{\mu}\right) \ , \tag{30b}$$

obey the Poincaré algebra, i.e.,

$$[P^{\mu}, P^{\nu}] = 0 \ , \tag{31}$$

$$[J^{\mu\nu}, P^{\sigma}] = i\left(g^{\mu\sigma}P^{\nu} - g^{\nu\sigma}P^{\mu}\right) \ , \tag{32}$$

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\mu\rho}J^{\nu\sigma} + g^{\nu\sigma}J^{\mu\rho} - g^{\mu\sigma}J^{\nu\rho} - g^{\nu\rho}J^{\mu\sigma}\right) \ . \tag{33}$$

It takes just a few more steps to demonstrate that Θ is a tensor. Indeed, the additional equal-time commutators $\left[\Theta^{ij}(x^0,\vec{x})\,,\,\Theta^{00}(x^0,\vec{y})\right]$ and $\left[\Theta^{ij}(x^0,\vec{x})\,,\,\Theta^{0k}(x^0,\vec{y})\right]$ can also be readily evaluated by using (28) and (15). These results and (29) can be collected into

$$\left[P^{\mu}\,,\,\Theta^{\alpha\beta}\right] = -i\,\partial^{\mu}\Theta^{\alpha\beta} \ , \tag{34a}$$

$$\left[J^{\mu\nu}\,,\,\Theta^{\alpha\beta}\right] = -i\left(x^{\nu}\partial^{\mu} - x^{\mu}\partial^{\nu}\right)\Theta^{\alpha\beta}$$
$$-i\left(\Theta^{\mu\alpha}g^{\nu\beta} + \Theta^{\mu\beta}g^{\nu\alpha} - \Theta^{\nu\alpha}g^{\mu\beta} - \Theta^{\nu\beta}g^{\mu\alpha}\right) \ , \tag{34b}$$

which are, respectively, the translation and rotation transformation laws to be obeyed by a second-rank tensor. The purported proof of relativistic invariance of the quantized Schwarz-Sen theory is now complete.

What remains to be done is to demonstrate that the Coulomb gauge formulation of the quantized Schwarz-Sen theory is in fact covariant. Since translations and ordinary rotations do not destroy the Coulomb gauge condition we concentrate on Lorentz boosts. By using (30), (28), (2b) and (13a) one arrives to

$$-i\left[J^{0k}\,,\,A_T^{a,i}\right] = (x^0\,\partial^k - x^k\,\partial^0)A_T^{a,i} - \epsilon_{ab}\epsilon^{klj}\frac{\partial^i\partial_l}{\nabla^2}A^b_{Tj} \ . \tag{35}$$

The term proportional to ϵ_{ab} signalizes that gauge potentials corresponding to different values of a get mixed by Lorentz boosts. This does not occur for ordinary rotations. Furthermore, the mixing term in (35) describes an operator gauge transformation, which, as one easily verifies, makes this commutator compatible with the transversality condition $\partial_i A_T^{a,i} = 0$. Hence, under Lorentz boosts, the field $A_T^{a,i}$ undergoes, besides the usual vector transformation, an operator gauge transformation which restores the Coulomb gauge in the new Lorentz frame.

As for the Nother's charge associated with the $SO(2)$ symmetry (4), it is straightforward to verify that it can be written as

$$Q = -\frac{1}{2}\int d^3x\epsilon^{jik}(\partial_j A^a_{Ti})A^a_{Tk} = \frac{1}{2}\int d^3x B^{ak}A^a_{Tk} \ . \tag{36}$$

Observe that Q is a SO(2) invariant Chern-Simons term. Thus, up to surface terms, it is gauge invariant. It is also metric independent and so its algebraic form also holds for curved spaces. The use of (13a) enables one to verify that Q indeed generates the infinitesimal $SO(2)$ rotations

$$[Q, A^b_{Tj}(y)] = -i\epsilon^{ba}A^a_{Tj}(y) \ . \tag{37}$$

Furthermore, in terms of the creation and anhilation operators of (23) the operator Q is found to read

$$Q = i\int d^3k(a_1^{\dagger}a_2 - a_2^{\dagger}a_1) \tag{38}$$

and becomes diagonal,

$$Q = \int d^3k(a_L^{\dagger}a_L - a_R^{\dagger}a_R) \ , \tag{39}$$

in the base of circularly polarized operators, defined by

$$a_R^\dagger = \frac{a_1^\dagger + i a_2^\dagger}{\sqrt{2}} \ , \tag{40a}$$

$$a_L^\dagger = \frac{a_1^\dagger - i a_2^\dagger}{\sqrt{2}} \ . \tag{40b}$$

From (39) one sees that, in a generic state, Q counts the number of left minus right polarized photons. It is easily checked that Q commutes with all the generators of the conformal group as should be the case for an internal symmetry generator.

The last part of this Section is dedicated to present the functional quantization of the Schwarz-Sen theory in the Coulomb gauge. Clearly, the constraints (8a) and (12a) can be used to eliminate the phase-space variables π_0^a, $A^{a,0}$ from the outset. On the other hand, the constraints $\Omega^a \approx 0$, $\chi^a \approx 0$ have vanishing Poisson brackets with those in the set $\{\Omega_{Ti}^a \approx 0\}$. This means that the Faddeev-Popov determinant split as follows

$$\det(\nabla^2)\ \det{}^{1/2}(\epsilon^{ab}\epsilon^{ijk}\partial_j) \ , \tag{41}$$

which after taking into account the functional relationship

$$\det{}^{1/2}(\epsilon_{ab}\epsilon_{ijk}\partial^j) = \det(\epsilon_{ijk}\partial^j) \ , \tag{42}$$

reduces to

$$\det(\nabla^2)\ \det(\epsilon^{ijk}\partial_j) \ . \tag{43}$$

Clearly, the first factor in (41) is the determinant of the matrix whose elements are $[\Omega^a(\vec{x}), \chi^b(\vec{y})]_P$, while the second is the determinant of the matrix whose elements are given at (9). Hence, for the model under analysis, the phase-space generating functional of Green functions ($\tilde{W}$) is given by

$$\begin{aligned} \tilde{W} \ =& \int [\mathcal{D}A^{a,i}]\,[\mathcal{D}\pi_i^a]\,\delta[\partial_i A^{a,i}]\,\delta[\partial^i \pi_i^a]\,\det(\nabla^2) \\ & \times \delta[\pi_i^a + \frac{1}{2}\epsilon^{ab}\epsilon_{ijk}\partial^j A^{b,k}]\ \det(\epsilon^{ijk}\partial_j)\,\mathrm{e}^{i\tilde{S}_{eff}} \ , \end{aligned} \tag{44}$$

where

$$\tilde{S}_{eff} \ = \int d^4x \left[\pi_i^a \dot{A}^{ia} - \frac{1}{2}(\vec{\nabla}\times\vec{A}^a)^2\right] \ , \tag{45}$$

is the corresponding effective action.

III. LOCAL DUALITY TRANSFORMATIONS FOR MAXWELL THEORY

As is well known, for the free Maxwell field, in the Coulomb gauge, the phase-space Green functions generating functional is given by[2]

$$W \ = \int [\mathcal{D}A^i]\,[\mathcal{D}\pi_i]\ \det(\nabla^2)\,\delta[\partial_i A^i]\,\delta[\partial^i \pi_i]\,\mathrm{e}^{iS_{eff}} \ , \tag{46}$$

where the effective action (S_{eff}) reads

$$S_{eff} \ = \int d^4x \left[\pi_i \dot{A}^i - \left(-\frac{1}{2}\pi^i\pi_i - \frac{1}{2}B^i B_i\right)\right] \ . \tag{47}$$

Here, $B^i = -\epsilon^{ijk}\partial_j A_k$ is the i-th component of the magnetic field, while π_i denotes the momentum canonically conjugate to A^i. As shown in Ref. [4], up to surface terms, S_{eff} remains invariant under the non-local duality transformations

[2]This section is mainly based on Ref. [8].

$$A^i \to A'^i \;=\; A^i + \delta_D A^i;\;\; \delta_D A^i \;=\; \theta\, \nabla^{-2}\, \epsilon^{ijk}\, \partial_j \pi_k \;\;, \tag{48a}$$

$$\pi_i \to \pi'_i \;=\; \pi_i + \delta_D \pi_i;\;\; \delta_D \pi_i \;=\; \theta\, \epsilon_{ijk}\, \partial^j A^k \;\;. \tag{48b}$$

The point we would like to stress now is that these transformations can be made local by introducing the auxiliary fields C^i, and their corresponding canonical conjugate momenta P_i, defined as follows

$$\nabla^2 C^i \;=\; \epsilon^{ijk}\, \partial_j \pi_k \;\;, \tag{49a}$$

$$P_i = \epsilon_{ijk}\, \partial^j A^k \;\;, \tag{49b}$$

$$\partial_i C^i \;=\; \partial^i P_i \;=\; 0 \;\;. \tag{49c}$$

In fact, from Eqs.(48) and (49) one verifies that

$$\delta_D A^i \;=\; \theta\, C^i, \quad \delta_D C^i \;=\; -\,\theta\, A^i, \tag{50a}$$

$$\delta_D \pi_i = \theta\, P_i, \quad \delta_D P_i = \,-\,\theta\, \pi_i \;\;. \tag{50b}$$

Our next task consists in reformulating the Maxwell theory in terms of the fields A^i, π_i, C^i and P_i. To this end, we first recall that

$$\left(\prod \delta[\;\;]\right)\, (\vec{\nabla} \times \vec{C})^2 \;=\; \left(\prod \delta[\;\;]\right)\, \pi_i \pi_i \;\;, \tag{51a}$$

$$\left(\prod \delta[\;\;]\right)\, \frac{1}{2} P_i \dot{C}^i \;=\; \left(\prod \delta[\;\;]\right)\, \frac{1}{2} \pi_i \dot{A}^i \;\;, \tag{51b}$$

where the following definition

$$\prod \delta[\;\;] \;\equiv\; \delta[\partial_i A^i]\, \delta[\partial^i \pi_i]\, \delta[\partial_i C^i] \delta[\partial^i P_i]\, \delta[C^i - \nabla^{-2} \epsilon^{ijk} \partial_j \pi_k] \delta[P_i - \epsilon_{ijk} \partial^j A^k] \;\;, \tag{52}$$

has been introduced. As consequence,

$$\left(\prod \delta[\;\;]\right)\, S_{eff} \;=\; \left(\prod \delta[\;\;]\right)\, \tilde{\tilde{S}}_{eff} \;\;, \tag{53}$$

where

$$\tilde{\tilde{S}}_{eff} \;\equiv\; \int d^4x \, \left\{ \frac{1}{2} \pi_i \dot{A}^i + \frac{1}{2} P_i \dot{C}^i - \frac{1}{2} \left[(\vec{\nabla} \times \vec{A})^2 \,+\, (\vec{\nabla} \times \vec{C})^2 \right] \right\}. \tag{54}$$

Correspondingly, the Coulomb gauge generating functional W of Maxwell theory can be cast as

$$\begin{aligned} W = & \int [\mathcal{D}A^i]\, [\mathcal{D}\pi_i]\, \delta[\partial_i A^i]\, \delta[\partial^i \pi_i] \\ & \times\, [\mathcal{D}C^i]\, [\mathcal{D}P_i]\, \delta[\partial_i C^i] \delta[\partial^i P_i]\, \det(\nabla^2) \\ & \times\, \delta[C^i - \nabla^{-2} \epsilon^{ijk} \partial_j \pi_k] \delta[P_i - \epsilon_{ijk} \partial^j A^k]\;\, \mathrm{e}^{i \tilde{\tilde{S}}_{eff}} \;\;. \end{aligned} \tag{55}$$

It is through this form of W that we shall make contact with the Schwarz-Sen model. Needless to say, W in (55) is, by construction, invariant under the set of local duality transformations (50). We learnt in this Section that, by means of an appropriate enlargement of the phase-space, one can incorporate duality as a local symmetry of sourceless electrodynamics.

IV. EQUIVALENCE OF THE MAXWELL AND SCHWARZ-SEN THEORIES

We have now at hand two duality symmetric theories. One is the Schwarz-Sen theory, whose phase space is spanned by the variables $A^{1,i}$, $A^{2,i}$, π^1_i and π^2_i. The other one is Maxwell theory, whose phase-space variables are A^i, C^i, π_i and P_i. We shall prove, in this Section, that these theories are quantum mechanically equivalent[3].

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[3]This Section is mainly based on Ref. [8]

The initial step toward this proof consists in identifying those variables whose behavior under infinitesimal duality transformations is the same. For the coordinates, the task is easy. Indeed, under infinitesimal duality transformations the Schwarz-Sen coordinates $A^{a,i}$ change as follows (see (4))

$$\delta_D A^{1,i} = \theta\, A^{2,i}, \;\; \delta_D A^{2,i} = -\,\theta\, A^{1,i} \;\; . \tag{56}$$

Therefore, if one sets

$$A^{1,i} = A^i \;\; , \tag{57}$$

one obtains, from (50a) and (56),

$$A^{2,i} = C^i \;\; . \tag{58}$$

The situation is slightly more involved for the momenta. By combining (8b), (56) and (57) one finds

$$\delta_D \pi^1_i = -\frac{1}{2}\epsilon_{ijk}\partial^j \delta_D A^{2,k} = \frac{\theta}{2}\epsilon_{ijk}\partial^j A^{1,k} = \frac{\theta}{2}\epsilon_{ijk}\partial^j A^k \;\; . \tag{59}$$

On the other hand, (48b) enables one to write

$$\frac{\theta}{2}\epsilon_{ijk}\partial^j A^k = \frac{1}{2}\delta_D \pi_i \;\; . \tag{60}$$

Therefore,

$$\pi^1_i = \frac{1}{2}\,\pi_i \;\; . \tag{61}$$

Through a similar calculation, which uses (49.b) instead of (48.b), one arrives to

$$\pi^2_i = \frac{1}{2}\,P_i \;\; . \tag{62}$$

We shall next take advantage of the identifications above to rewrite the generating functional of Maxwell theory in terms of the Schwarz-Sen variables. This change of integration variables in (55) leads to

$$\begin{aligned} W = & \int [\mathcal{D}A^{a,i}]\,[\mathcal{D}\pi^a_i]\,\delta[\partial_i A^{a,i}]\,\delta[\partial^i \pi^a_i]\,\det(\nabla^2) \\ & \times\; \delta[A^{2,i} - 2\,\nabla^{-2}\epsilon^{ijk}\partial_j \pi^1_k]\delta[2\,\pi^2_i - \epsilon_{ijk}\partial^j A^{1,k}] \\ & \times\; e^{i\tilde{S}_{eff}} \;\; . \end{aligned} \tag{63}$$

We emphasize that, when written in terms of the Schwarz-Sen variables, the Maxwell action ($\tilde{\tilde{S}}_{eff}$) becomes the Schwarz-Sen action ($\tilde{S}_{eff}$). However, since the integration measure in (63) does not appear to be that in (44), a few more algebraic manipulations will be required to establish the equivalence between these theories. Let L^{ik} be the differential operator

$$L^{ik} \equiv \epsilon^{ijk}\partial_j \;\; . \tag{64}$$

Then,

$$\delta[\partial_i A^{2,i}]\,A^{2,i} = \delta[\partial_i A^{2,i}]\,L^{ik}\nabla^{-2}L_{km}A^{2,m} \;\; , \tag{65}$$

which, in particular, implies that

$$\begin{aligned} & \delta[\partial_i A^{2,i}]\,\delta[A^{2,i} - 2\,\epsilon^{ijk}\,\nabla^{-2}\,\partial_j \pi^1_k] \\ = & \;\delta[\partial_i A^{2,i}]\det{}^{-1}\left(-\epsilon^{ijk}\partial_j\nabla^{-2}\right)\,\delta[\epsilon_{klm}\partial^l A^{2,m} + 2\,\pi^1_k] \;\; . \end{aligned} \tag{66}$$

On the other hand, from the eigenvalue equation

$$\delta[\partial_j \phi^j]\,L^{ik}\nabla^{-2}L_{km}\,\phi^m = -\,\delta[\partial_j \phi^j]\,\phi^i \;\; , \tag{67}$$

one learns that

$$\det\left(L^{ik}\nabla^{-2}L_{km}\right) \;=\; constant \;\;, \tag{68}$$

or, equivalently,

$$\det{}^{-1}\left(-\epsilon^{ijk}\partial_j\nabla^{-2}\right) \;=\; constant \times \det\left(\epsilon_{klm}\partial^l\right) \;\;. \tag{69}$$

By going back with (66) and (69) into (63) one finds that

$$\begin{aligned} W &= constant \times \int [\mathcal{D}A^{a,i}]\,[\mathcal{D}\pi_i^a]\,\delta[\partial_i A^{a,i}]\,\delta[\partial^i\pi_i^a]\;\det(\nabla^2) \\ &\times\,\delta[\pi_i^a + \frac{1}{2}\epsilon^{ab}\epsilon_{ijk}\partial^j A^{b,k}]\,\det(\epsilon^{ijk}\partial_j) \\ &\times\;\exp\left\{ i\int d^4x\,\left[\pi_i^a A^{a,i} \;-\; \frac{1}{2}\,\left(\vec{\nabla}\times\vec{A}^a\right)^2\right]\right\} \\ &= constant \times \tilde{W} \;\;. \end{aligned} \tag{70}$$

The purported proof of equivalence between the Maxwell and the Schwarz-Sen theories is now complete.

V. CONCLUSIONS

We started in this work by canonically quantizing the Schwarz-Sen theory in the Coulomb gauge. The resulting theory turned out to be physically sensible. The Lagrangian density defining the theory is not covariant but, nevertheless, we were able of constructing a set charges verifying the Poincaré algebra.

Later, we showed that the phase space associated with the Coulomb gauge formulation of Maxwell theory can be conveniently enlarged in order to accommodate duality as a local symmetry.

By analyzing the behavior under duality transformation, we were able of identifying the phase-space variables of the Maxwell theory with those of the Schwarz-Sen theory. It was then possible to demonstrate that the corresponding Green functions generating functionals were the same.

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